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On the Lipschitz Regularity of Solutions of a Minimum Problem with Free Boundary

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Abstract

In this article under assumption of "small" density for negativity set, we prove local Lipschitz regularity for the one phase minimization problem with free boundary for the functional

$$\mathcal{E}_p(v, \Omega) = \int_{\Omega} |\nabla v|^p + \lambda_1^p \chi_{\{u \leq 0\}} + \lambda_2^p \chi_{\{u > 0\}}, \quad 1 < p < \infty,$$

where λ_1, λ_2 are positive constants so that $\Lambda = \lambda_1^p - \lambda_2^p < 0$, χ_D is the characteristic function of set D , $\Omega \subset \mathbf{R}^n$ is (smooth) domain and minimum is taken over a suitable subspace of $W^{1,p}(\Omega)$.

1 Introduction

Let $\mathcal{K}_g = \{v \in W^{1,p}(\Omega) : v - g \in W_0^{1,p}(\Omega)\}$ for prescribed smooth function g $\Omega \subset \mathbf{R}^n$ and consider the energy minimization problem,

$$\mathcal{E}_p(u, \Omega) = \inf_{v \in \mathcal{K}_g} \mathcal{E}_p(v, \Omega), \quad 1 < p < \infty \quad (1)$$

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with

$$\mathcal{E}_p(u, \Omega) = \int_{\Omega} |\nabla u|^p + \lambda_1^p \chi_{\{u \leq 0\}} + \lambda_2^p \chi_{\{u > 0\}}.$$

Here $\Omega \subset \mathbf{R}^n$ is bounded and smooth domain, λ_1, λ_2 are positive constants so that $\Lambda = \lambda_1^p - \lambda_2^p < 0$, χ_M is the characteristic function of the set $M \in \mathbf{R}^n$, i.e.

$$\chi_M = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{if } x \notin M. \end{cases}$$

The minimizer u is expected to verify to the following overdetermined problem

$$\Delta_p u = 0 \quad \text{in } u \neq 0, \quad |\nabla u^+|^p - |\nabla u^-|^p = c \quad \text{on } \partial\{u > 0\}, \quad u = g \quad \text{on } \partial\Omega, \quad (2)$$

where the u^+, u^- are respectively the positive and negative parts of u , c is a positive constant and the boundary data g is not necessarily nonnegative. This problem, usually termed Bernoulli-type problem, models for example cavitation flow of one or two perfect fluids, or equilibrium configuration for heat or electrostatic energy optimization. Weak solutions of problem 2 can be obtained by minimizing \mathcal{E}_p , (see theorem 2) and our objective here is to analyze the regularity of those solutions u .

Since u has a jump along the *free boundary* $\Gamma = \partial\{u > 0\}$, the best expected regularity for u is Lipschitz continuity. In the classical case $p = 2$, corresponding to usual Laplacian, this is proved in [ACF], and in [DP] for any $1 < p < \infty$ and $u^- \equiv 0$. The main complexity, in attacking the Lipschitz regularity for general case, is the lack of monotonicity formulas, firstly introduced in [ACF], and subsequently developed in [CJK], [CKS]. However we can still prove that $u \in C_{loc}^{0,1}$ if the negativity set $\Omega^-(u) = \{u < 0\}$ is reasonably small. The $C^{0,1}$ estimate plays vital role in establishing $C^{1,\alpha}$ regularity of free boundary near flat points. However here we solely focus upon proving local $C^{0,1}$ estimate for solutions. The present study has been inspired by a recent work [KKS] and by [LS], where similar result is proven for another overdetermined problem:

$$F(D^2 u) = \chi_{\{D\}} \quad \text{in } B_1, \quad u = |\nabla u| = 0 \quad \text{in } B_1 \setminus D, \quad (3)$$

for a certain class of uniformly elliptic operators F . We observe here that unlike to (3) we don't have a pde, to which solutions u of (1) would verify in Ω .

2 Preliminaries

The following notations are used throughout the paper: $\Omega \subset \mathbf{R}^n$ is a smooth and bounded domain, g is a smooth function defined on some neighborhood of $\partial\Omega$, $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ are the usual Sobolev spaces, $B_R(y) = \{x \in \mathbf{R}^n : |x - y| < R\}$, $B_R = B_R(0)$, u^\pm are respectively the positive and the negative parts of u , $\chi_{\{D\}}$ the characteristic function of D , $\Gamma = \partial\{u > 0\}$ free boundary. Let λ_1 and λ_2 be two positive constants so that $\Lambda = \lambda_1^p - \lambda_2^p < 0$ where $1 < p < \infty$. Consider functional

$$\mathcal{E}_p(u, \Omega) = \int_{\Omega} |\nabla u|^p + \lambda_1^p \chi_{\{u \leq 0\}} + \lambda_2^p \chi_{\{u > 0\}}.$$

In what follows we denote by $\lambda(u)$ the following function:

$$\lambda(u) = \begin{cases} \lambda_1^p & \text{if } u \leq 0 \\ \lambda_2^p & \text{if } u > 0. \end{cases}$$

As in the classical paper [ACF] we define $\lambda(0) = \lambda_1^p$ if $\Lambda < 0$ and $\lambda(0) = \lambda_2^p$ if $\Lambda > 0$. For brevity we focus on the case $\Lambda < 0$. Existence of solutions to (1) easily follows from the lower semicontinuity of \mathcal{E}_p as in [ACF].

Theorem 1 *Let u be a (local) minimizer of \mathcal{E}_p . Then u is bounded.*

Proof: First let us observe that

$$\int_{\Omega} |\nabla u|^p + \lambda_1^p \chi_{\{u \leq 0\}} + \lambda_2^p \chi_{\{u > 0\}} = \int_{\Omega} |\nabla u|^p + \Lambda \chi_{\{u \leq 0\}} + \lambda_2^p \text{meas} D. \quad (4)$$

For given $D \subset \Omega$ let us consider the functional $I_0(u, D) = \int_D |\nabla u|^p + \Lambda \chi_{\{u \leq 0\}}$. If u is a minimizer of $\mathcal{E}_p(u, D)$ then it is also a minimizer of $I_0(u, D)$ and vice versa since the difference between I_0 and \mathcal{E}_p is a constant for given domain D .

Now take $u_\varepsilon = u + \varepsilon \min(M - u, 0)$, where $M = \sup g > 0$ and ε is a small positive number. Then taking $D = \Omega$ and testing u against u_ε we get

$$\int_{\Omega} |\nabla u|^p + \Lambda \chi_{\{u \leq 0\}} \leq \int_{\Omega} |\nabla u_\varepsilon|^p + \Lambda \chi_{\{u_\varepsilon \leq 0\}}$$

Note that u and u_ε are different on the set $\{u > M\}$, therefore last inequality becomes

$$\int_{\Omega \cap \{u > M\}} |\nabla u|^p \leq \int_{\Omega \cap \{u > M\}} |\nabla u|^p (1 - \varepsilon)^p + \Lambda \chi_{\{u_\varepsilon \leq 0\}}$$

which is a contradiction since $\Lambda < 0$ and hence $u \leq M$. Now take $u_\varepsilon = u - \min(u - m, 0)$ where $m = \inf u < 0$ and ε is a positive number. Again since u is a minimizer we have

$$\int_{\Omega} |\nabla u|^p + \Lambda \chi_{\{u \leq 0\}} \leq \int_{\Omega} |\nabla u_\varepsilon|^p + \Lambda \chi_{\{u_\varepsilon \leq 0\}}.$$

On the set $\{u < m\}$, where u and u_ε are different we have that

$$\int_{\Omega \cap \{u < m\}} |\nabla u|^p + \Lambda \chi_{\{u < m\}} \leq \int_{\Omega \cap \{u < m\}} |\nabla u_\varepsilon|^p (1 - \varepsilon)^p + \Lambda \chi_{\{u \leq -\frac{\varepsilon m}{1 - \varepsilon}\}}.$$

Note that $-\frac{\varepsilon m}{1 - \varepsilon} > 0$ and therefore we get that

$$\begin{aligned} \int_{\Omega \cap \{u < m\}} |\nabla u|^p &\leq \int_{\Omega \cap \{u < m\}} |\nabla u_\varepsilon|^p (1 - \varepsilon)^p + \Lambda \left[\chi_{\{u \leq -\frac{\varepsilon m}{1 - \varepsilon}\}} - \chi_{\{u < m\}} \right] \\ &= \int_{\Omega \cap \{u < m\}} |\nabla u_\varepsilon|^p (1 - \varepsilon)^p. \end{aligned}$$

This implies that $m \leq u$. □

Theorem 2 $u \in C_{loc}^\alpha(\Omega)$.

Proof: Let $B_R(y) \subset \Omega$ and w be the solution to the folloing Dirichlet problem

$$\Delta_p w = 0 \quad \text{in } B_R(y), \quad w = u \quad \text{on } \partial B_R(y). \quad (5)$$

Then we have that

$$\int_{B_R(y)} |\nabla u|^p + \lambda(u) \leq \int_{B_R(y)} |\nabla w|^p + \lambda(w)$$

where $\lambda(u) = \lambda_1^p \chi_{\{u \leq 0\}} + \lambda_2^p \chi_{\{u > 0\}}$. Note that we also have

$$\int_{B_R(y)} |\nabla u|^p \geq \int_{B_R(y)} |\nabla w|^p.$$

Since $\lambda(u)$ is bounded it implies that

$$\int_{B_R(y)} [|\nabla u(x)|^p - |\nabla v(x)|^p] dx \leq CR^n \quad (6)$$

Furthermore one has from [DP]

$$\int_{B_R(y)} [|\nabla u|^p - |\nabla v|^p] \geq \begin{cases} c \left(\int_{B_R(y)} |\nabla(u-v)|^p \right)^{2/p} \left(\int_{B_R(y)} |\nabla u|^p \right)^{1-2/p}, & 1 < p \leq 2, \\ c \int_{B_R(y)} |\nabla(u-v)|^p, & 2 \leq p < \infty. \end{cases} \quad (7)$$

which together with (6) implies that

$$\int_{B_R(y)} |\nabla(u-w)|^p \leq \begin{cases} C\lambda_+^{p^2/2} R^{np/2} \left(\int_{B_R(y)} |\nabla u|^p \right)^{1-p/2}, & 1 < p \leq 2 \\ C\lambda_+^p R^n, & 2 \leq p < \infty. \end{cases} \quad (8)$$

Recall that from the gradient estimates for harmonic functions we have that

$$\sup_{B_{R/2}(y)} |\nabla w| \leq C \frac{\sup_{\Omega} |u|}{R}$$

Now for small R and $p > 2$ we have

$$\begin{aligned} \int_{B_{R/2}(y)} |\nabla u|^2 &\leq C \int_{B_{R/2}(y)} |\nabla(u-w)|^p + C \int_{B_{R/2}(y)} |\nabla w|^p \\ &\leq C \int_{B_{R/2}(y)} |\nabla(u-w)|^p + CR^{n-p}. \end{aligned} \quad (9)$$

Then combining (8) and (9) as in [DP] the result follows. \square

Corollary 1 *u is p -subharmonic*

Proof: We first note that if v verifies to

$$\Delta_p v = 0 \quad \text{in } B_R(y), \quad v = u \quad \text{on } \partial B_R(y).$$

where $B_R(y) \subset \Omega$, then testing u against $\min(u, v)$ we have we find that

$$\int_{B_R(y)} [|\nabla u(x)|^p - |\nabla \min(u(x), v(x))|^p] dx \leq \Lambda \int_{B_R \cap \{u > 0 \geq v\}} 1 dx.$$

Since u is Hölder continuous, the set $\{u > v\}$ is open and we can apply (7) to infer that

$$\int_{B_R(y)} [|\nabla u|^p - |\nabla \min(u, v)|^p] > 0.$$

However $\Lambda < 0$, which yields $\max(u - v, 0) = 0$ in B_R , that is $u \leq v$ in B_R . Hence u is p -subharmonic in Ω . \square

Before proceeding further we summarize some basic properties of solutions to (1).

Theorem 3 *Let u be the solution to (1). Then*

- $\Delta_p u = 0$ in $[\{u > 0\} \cup \{u < 0\}] \cap \Omega$,
- $\Delta_p u \geq 0$ in Ω ,
- $\lim_{\varepsilon \downarrow 0} \int_{\partial\{u < -\varepsilon\}} ((p-1)|\nabla u|^p - \lambda_1^p) \nu \cdot \eta + \lim_{\delta \downarrow 0} \int_{\partial\{u > \delta\}} ((p-1)|\nabla u|^p - \lambda_2^p) \nu \cdot \eta = 0$ for any $\eta \in C_0^1(\Omega, \mathbf{R}^n)$ provided $\text{meas}\{u = 0\} = 0$.

The proof follows precisely as in [ACF].

3 Main result

In this section we assume that $\lambda_1 = 0$, since introducing $\lambda_0^p = \lambda_2^p - \lambda_1^p = -\Lambda > 0$ we can consider a new functional

$$\int_{\Omega} |\nabla u|^p + \lambda_0^p \chi_{\{u > 0\}} = \mathcal{E}_p(u, \Omega) - \lambda_1^p \text{meas} \Omega$$

Therefore we identify $\mathcal{E}_p(u, \Omega)$ with $\int_{\Omega} |\nabla u|^p + \lambda^p \chi_{\{u > 0\}}$ for some positive constant λ . Next we define the main class of functions that we are going to work with.

Definition 1 *Let z be a fixed point and $0 < r < 1$. u is said to be of class $\mathcal{Q}_r(z, M)$ if*

- (i) u is a local minimizer of \mathcal{E}_p in $B_r(z)$,
- (ii) $\sup_{B_r(z)} |u| \leq M$,
- (iii) $z \in \partial\{u > 0\}$.

Let

$$\Theta(x_0, r) = \frac{\text{meas}(\{u < 0\} \cap B_r)}{\text{meas} B_r}, x_0 \in \partial\{u > 0\}$$

Theorem 4 *Let $u \in \mathcal{Q}_1(x_0, M)$. There exists a positive universal constant $C > 0$ such that*

$$|u(x)| \leq \frac{2M}{C} |x|$$

provided $\Theta(x_0, r) \leq C$ for all $0 < r < 1$.

Proof: Without loss of generality we may assume $x_0 = 0$. It is enough to prove that

$$\sup_{B_{2^{-(k+1)}}} |u(x)| \leq \max \left\{ \frac{M}{C2^k}, \frac{S(k)}{2}, \dots, \frac{S(k-m)}{2^{m+1}}, \dots, \frac{S(0)}{2^{k+1}} \right\} \quad (10)$$

where $S(k) = \sup_{B_{2^{-k}}} |u|$. Assume a contradiction. Then there are integers $k_j, j = 1, 2, \dots$ so that

$$\sup_{B_{2^{-(k_j+1)}}} |u_j(x)| > \max \left\{ \frac{jM}{2^{k_j}}, \frac{S_j(k_j)}{2}, \dots, \frac{S_j(k_j-m)}{2^{m+1}}, \dots, \frac{S_j(0)}{2^{k_j+1}} \right\} \quad (11)$$

and

$$\Theta(0, 2^{-k_j}) \leq \frac{1}{j} \rightarrow 0. \quad (12)$$

Here

$$S_j(k_j - m) = \sup_{B_{2^{-(k_j-m)}}} |u_j|, m = 0, 1, 2, \dots, k_j.$$

$u_j \in \mathcal{Q}_1(z, M)$. Observe that $|u_j| \leq M$ implies $k_j \rightarrow \infty$.

Consider auxiliary function v_j defined as

$$v_j(x) = \frac{u_j(x2^{-k_j})}{S_j(k_j + 1)}$$

We start by proving $W^{1,p}$ estimates for v_j . Set $\sigma_j = 2^{-k_j} S_j^{-1}(k_j + 1)$. Note that by (11) $\sigma_j \leq j^{-1} \rightarrow 0$. For fixed $R_0 > 0$ we have

$$\begin{aligned} \int_{B_{R_0}} |\nabla v_j(x)|^p dx &= \sigma_j^p \int_{B_{R_0}} |\nabla u_j(x 2^{-k_j})|^p dx \\ &= \sigma_j^p 2^{n k_j} \int_{B_{R_0 2^{-k_j}}} |\nabla u_j(y)|^p dy \end{aligned} \quad (13)$$

Let $\rho > 0$ and φ is the standard cut-off function of B_ρ . Then if $\eta = \varphi^p u_j^+$ is a admissible test function and (ii) yields

$$\int_{B_\rho} |\nabla u_j^+|^{p-2} \nabla u_j^+ \nabla \eta \leq 0.$$

Rearranging the terms and after using Hölder inequality we get

$$\begin{aligned} \int_{B_\rho} \varphi^p |\nabla u_j^+|^p &\leq p \int_{B_\rho} |\nabla u_j^+|^{p-1} \varphi^{p-1} |\nabla \varphi| u_j^+ dx \leq \\ &= p \left(\int_{B_\rho} |\nabla \varphi|^p (u_j^+)^p dx \right)^{\frac{1}{p}} \left(\int_{B_\rho} |\nabla u_j^+|^p \varphi^p dx \right)^{1-\frac{1}{p}} \end{aligned} \quad (14)$$

So we get Caccioppoli's inequality

$$\int_{B_{\rho/2}} |\nabla u_j^+|^p \leq \frac{c}{\rho^p} \int_{B_\rho} (u_j^+)^p \leq c \rho^{n-p} \left(\sup_{B_\rho} |u_j| \right)^p. \quad (15)$$

Let us take $\frac{\rho}{2} = \frac{R_0}{2^{k_j}}$ in the last inequality,

$$\int_{B_{R_0 2^{-k_j}}} |\nabla u_j^+|^p \leq c \left(\frac{2R_0}{2^{k_j}} \right)^{n-p} \left(\sup_{B_{\frac{2R_0}{2^{k_j}}}} |u_j| \right)^p.$$

Choose $R_0 = 2^{l-1}$ for l , fixed integer $l < k_j$ we have then

$$\begin{aligned} \int_{2^{l-1}} |\nabla v_j^+|^p &\leq c \left[\frac{2^{-k_j}}{S_j(k_j + 1)} \right]^p 2^{n k_j} 2^{(l-k_j)(n-p)} \left(\sup_{B_{2^{l-k_j}}} |u_j| \right)^p \leq \\ &\leq c \left[\frac{2^{-k_j}}{S_j(k_j + 1)} \right]^p 2^{n k_j} 2^{(l-k_j)(n-p)} \left(2^{l+1} S_j(k_j + 1) \right)^p = \\ &= 2^{ln+p}, \end{aligned} \quad (16)$$

where the second inequality follows from (11). Therefore $\|\nabla v_j\|_{L^p}$ is locally bounded implying local uniform $W^{1,p}$ estimates for v_j for j large.

If $p > n$ then the Sobolev imbedding theorem implies uniform local C^α estimate for v_j , for j large. Suppose $1 < p \leq n$. Consider the scaled energy functional

$$\mathcal{E}_j(v, D) = \int_D |\nabla v|^p + \sigma_j^p \lambda^p \chi_{\{v>0\}} \quad (17)$$

First let us observe that a simple calculation gives

$$\mathcal{E}_j(v_j, B_{R_0}) = \sigma_j^p 2^{nk_j} \mathcal{E}_p(u_j, B_{R_0 2^{-k_j}}). \quad (18)$$

Therefore v_j is a solution to

$$\mathcal{E}_j(v_j) = \inf_{v \in \mathcal{K}_j} \mathcal{E}_j(v)$$

$$\mathcal{K}_j = \{v \in W^{1,p}(B_{2^{k_j}}), v - v_j \in W_0^{1,p}(B_{2^{k_j}})\}.$$

Applying Theorem 1 to v_j we have uniform C_{loc}^α estimate. Using uniform $W^{1,p}$ and C_{loc}^α estimates we have at least for a subsequence that

$$v_j \rightarrow v_\infty, \text{ in } W^{1,p}(B_2) \cap C^\alpha(B_2) \quad (19)$$

Now we claim that v_∞ is a local minimizer of $D_p(v) = \int |\nabla v|^p$. Indeed, for any $\varphi \in C_0^\infty(B_1)$ we have

$$\int_{B_1} |\nabla v_j|^p + \sigma_j^p \lambda^p \chi_{\{v_j>0\}} \leq \int_{B_1} |\nabla(v_j + \varphi)|^p + \sigma_j^p \lambda^p \chi_{\{v_j+\varphi>0\}}$$

By (19) we have

$$\begin{aligned} \int_{B_1} |\nabla v_j|^p &\rightarrow \int_{B_1} |\nabla v_\infty|^p \\ \int_{B_1} |\nabla(v_j + \varphi)|^p &\rightarrow \int_{B_1} |\nabla(v_\infty + \varphi)|^p. \end{aligned}$$

Since also $\sigma_j \leq \frac{1}{j}$, we get

$$\sigma_j^p \int_{B_1} \lambda^p \chi_{\{v_j>0\}} \rightarrow 0,$$

$$\sigma_j^p \int_{B_1} \lambda^p \chi_{\{v_j + \varphi > 0\}} \rightarrow 0.$$

Hence we conclude that

$$\int_{B_1} |\nabla v_\infty|^p \leq \int_{B_1} |\nabla(v_\infty + \varphi)|^p.$$

In view of C^α regularity this yields that v_∞ is a local minimizer for $D_p(v)$ in B_1 .

From definition of v_j and (12) we conclude:

- $0 \leq v_\infty \leq 2$, in B_1
- $\Delta_p v_\infty = 0$, in B_1
- $v_\infty(0) = 0$
- $\sup_{B_{\frac{1}{2}}} |v_\infty| = 1$

which contradicts to the strong maximum principle. \square

Corollary 2 *Assume that $\Theta_r(z, r) \leq C$ for all $z \in B_{1/2} \cap \Gamma$, then $u \in \mathcal{Q}_1(0, M)$ is Lipschitz in $B_{1/4}$.*

Proof: Let $u(x) > 0$ and $d(x) = \text{dist}(x, \partial\{u > 0\})$. Let $z \in \partial\{u > 0\}$ so that $d(x) = |x - z|$. Then $u(x) \leq 2MC^{-1}d(x)$. By Harnack's inequality $u \leq 2cMC^{-1}d(x)$ in $B_{d(x)/2}$. Consider $v(y) = \frac{u(x+d(x)y)}{d(x)}$. Then

$$\Delta_p v = 0, \quad \text{in } B_1, \quad 0 \leq v(y) \leq 2cMC^{-1} \quad \text{in } B_{1/2}.$$

Then from local gradient estimate $|\nabla v(0)| \leq C(n, p, M, C)$. \square

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